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A note on the stochastic lattice-gas model

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Abstract. It is proved, without imposing translation invariance, that the only stationary states for the infinite two-dimensional stochastic lattice-gas (or binary alloy) model are the canonical Gibbs states, and that these dynamics describe a strong return to equilibrium.

1. Introduction

A number of very interesting dynamical processes such as the cooling of a gas and the consequent transition to a mixed gas–liquid system or to the metastable state of an undercooled gas could not, until now, be described by realistic models involving the intermolecular forces. A very strongly similar process is that of the quenching of a binary alloy.

Simulated models have therefore been proposed in the hope that they might reveal, at least in a qualitative way, all physically important phenomena. In this note we study the model, originally introduced by Kawasaki (1966) and, later, by Spitzer (1970) in the mathematical literature. The model describes jumps of individual particles from one lattice site to a neighbouring site, with certain transition probabilities depending on the external temperature and the energy change resulting from the jump, in such a way that the detailed balance condition is satisfied. The interaction is a negative pair interaction, which need not be translation invariant.

We shall study the stationary states and the return to equilibrium, described by the infinite two-dimensional model. We do not impose translation invariance of the stationary state *a priori* but nevertheless prove that a state is stationary if and only if it is a canonical Gibbs state. Our result therefore completes Georgii's analysis (1979), which derived the same result imposing translation invariance. (In addition, Georgii proved the strict decrease of the specific free energy density). It should be remarked that both the present result and that of Georgii are based on a technique designed by Moulin-Ollagnier and Pinchon (1977) and by Holley and Stroock (1977), for similar studies of the Glauber model. Though partly an adaptation of those methods, some additional work is needed here to preclude the possibility of vanishing local probabilities.

Secondly, we prove that any state which is absolutely continuous with respect to a stationary state, converges for large times and in the norm topology, to that stationary state, thus showing a strong 'return' to equilibrium.

Conclusive rigorous results on the 'approach' to equilibrium have not yet been reached. Nevertheless, there now exists a vast literature on these matters, obtained via Monte-Carlo simulations. See for instance the series of papers by Kalos *et al* (1978)

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and various co-workers or Binder's book (1979). Most of these papers are concerned with the analysis of the structure function $S(k, t)$, or the growth of clusters.

We shall formulate our results in terms of the lattice-gas language. As stressed in Holley and Stroock (1977) the proof of the result on the stationary states does not work in three dimensions. The result on the return to equilibrium is independent of dimension.

We start by introducing the appropriate definitions and notation.

2. The model

2.1. *The configuration space*

As is usual in lattice-gas models we take as configuration space X the set $\{0, 1\}^{\mathbb{Z}^2}$ with the product topology. A configuration x is an element of X, taking the value $x(k)$ in the lattice site k.

In general, Λ will denote a finite subset of $\mathbb{Z}^2$, and $\tilde{\Lambda}$ its complement. Then X_Λ, $X_{\tilde{\Lambda}}$ will represent the configuration spaces for Λ and $\tilde{\Lambda}$ respectively. Given $x \in X$, x_Λ will denote its projection on X_Λ. Supposing $a \in X_{\Lambda_1}$ and $b \in X_{\Lambda_2}$ with $\Lambda_1 \cap \Lambda_2 = \phi$ we write a, b for the joint configuration in $X_{\Lambda_1} \cup \Lambda_{\Lambda_2}$. The notation 1^a_Λ will stand for the characteristic function of the event $x_\Lambda = a$. 0 denotes the empty configuration, i the fully occupied configuration.

When defining, in § 2.4, the dynamics, we need the following transformations of X: if $x \in X$; $k, l \in \mathbb{Z}^2$ we denote by x_{kl} the configuration with

$$\begin{aligned} x_{kl}(m) &= x(m) && \text{if } m \neq k, l \\ &= x(l) && \text{if } m = k \\ &= x(k) && \text{if } m = l \end{aligned}$$

and by x_k, the configuration with

$$\begin{aligned} x_k(m) &= x(m) && \text{if } m \neq k \\ &= 1 - x(k) && \text{if } m = k. \end{aligned}$$

Besides X, there is a family of reduced configuration spaces $X^{n,m}$ $(n, m \in \mathbb{N})$ involved. $X^{n,m}$ is the set of configurations with at most $(n-1)$ sites occupied or at most $(m-1)$ sites vacant.

Let us write $|x| \equiv \Sigma_{k \in \mathbb{Z}^2} x(k)$ and $|\bar{x}| \equiv \Sigma_{k \in \mathbb{Z}^2} (1 - x(k))$. As the maps $x \to |x|$ and $x \to |\bar{x}|$ are lower semicontinuous, we obtain that $X^{n,m}$ is compact for the relative topology.

Occasionally, when $a \in X_\Lambda$, we shall use the notation $|a|$ for the number $\Sigma_{k \in \Lambda} a(k)$, too.

$\chi^n (n \in \mathbb{N})$ will denote the characteristic function for the event $|x| = n$.

2.2. *The algebra and its states*

A will be the algebra of all complex continuous functions on X, endowed with the supremum-norm topology. A_Λ then denotes the local algebra for the region Λ, D the algebra of functions depending on finitely many coordinates only.

Next we denote by Ω the set of states on A, or, equivalently, the (regular Borel) probability measures on X. Similarly $\Omega^{n,m}$ is the set of probability measures on $X^{n,m}$. δ_0 and δ_i are the Dirac measures concentrated on the configurations 0 and i respectively.

If then $\omega \in \Omega$, ω_Λ will be its restriction to A_Λ, so that we shall write $\omega_\Lambda(a)$ for the probability of the event that $x_\Lambda = a$, $(a \in x_\Lambda)$.

Again for $\omega \in \Omega$, we introduce measures $\omega_{\tilde\Lambda}(x_\Lambda, \cdot\,)$ on $X_{\tilde\Lambda}$, for $x_\Lambda \in X_\Lambda$, through the formula

$$\omega(f) = \sum_{x_\Lambda \in X_\Lambda} \int \omega_{\tilde\Lambda}(x_\Lambda; \mathrm{d}x_{\tilde\Lambda})\, f(x).$$

for all f in A.

If E_Λ denotes the σ-algebra of all events, invariant under permutations of sites in Λ, we write $\omega(x_\Lambda = a/E_\Lambda)$ for the conditional probability of the event $x_\Lambda = a$, with respect to the measure ω and the σ-algebra E_Λ.

Due to the specific nature of the time evolution we now introduce canonical Gibbs states rather than Gibbs states.

2.3. *Canonical Gibbs states*

Let H_Λ, the local Hamiltonian, which is of the usual negative pair potential type be

$$H_\Lambda(x) = -J \sum_{k,l \in \Lambda}{}' x(k)x(l) - J \sum_{k \in \Lambda, l \notin \Lambda}{}' x(k)x(l).$$

Using standard notation, the primed sum indicates summation over nearest neighbours with each pair counted once only.

Definition (Georgii 1979). A state ω in Ω is a canonical Gibbs state at inverse temperature β, (and for the interaction H_Λ) iff for all finite Λ, all $a \in X_\Lambda$

$$\omega(x_\Lambda = a/E_\Lambda) = \frac{\exp[-\beta H_\Lambda(a, \cdot\,)]}{\sum_{b \in X_\Lambda; |b| = |x_\Lambda|} \exp[-\beta H_\Lambda(b, \cdot\,)]} \qquad \text{if } |a| = |x_\Lambda|$$

ω almost everywhere, and zero when $|a| \neq |x_\Lambda|$. □

The relation between these canonical Gibbs states and the more familiar Gibbs states is investigated by Georgii (1979).

2.4. *The stochastic lattice-gas dynamics*

Define, for any pair $\{k, l\}$ of lattice sites, the jump rate $c(k, l; \cdot\,)$

$$c(k, l, x) = \frac{\exp[-\beta H_{\{k,l\}}(x_{kl})]}{\exp[-\beta H_{\{k,l\}}(x_{kl})] + \exp[-\beta H_{\{k,l\}}(x)]},$$

a choice satisfying the detailed balance condition

$$c(k, l; x) \exp[-\beta H_{\{k,l\}}(x)] = c(k, l; x_{kl}) \exp[-\beta H_{\{k,l\}}(x_{kl})].$$

Consider the map $\mathscr{L}: D \to D$ defined by

$$(\mathscr{L}f)(x) = \sum_{k,l}{}' c(k, l; x)[f(x_{kl}) - f(x)].$$

Theorem 4.2 in Liggett (1971) shows that the closure of $\mathscr{L}$, also denoted by $\mathscr{L}$, generates a dynamical, or Markov semigroup, on A. Although only nearest-neighbour pairs are present in the definition of $\mathscr{L}$, more general $c(k, l; x)$ will be needed in lemma 3.4. We think of $c(k, l; x)$ as being the rate of transition from the configuration x to the configuration x_{kl}. It is determined solely by the values the configurations x and x_{kl} take in the sites k, l and their nearest neighbours.

3. Stationary states

In this section we determine the set of stationary states for the semigroup constructed in § 2.4. Since the process preserves the particle number and since we do not impose translation invariance of the state, it might be expected to find stationary states amongst the $\Omega^{m,n}$ class, and different from convex combinations of δ_0 and δ_i. We show, however, that this is precluded.

As remarked in the introduction, the result is proved using the techniques of Holley and Stroock (1977), the main difference being caused by the fact that in our model no creation or destruction of particles is allowed for. It was this creation and destruction which related $\omega_\Lambda(a)$ to $\omega_\Lambda(b)$ for $|a| \neq |b|$, thus enabling one to exclude $\omega_\Lambda(a) = 0$ when ω was stationary for the stochastic Ising ferromagnet. This is not possible *a priori* in the present model. It should be remarked that the proof can be adapted so as to incorporate non-translation invariant interactions too.

3.1. Lemma

If ω is T_t stationary, and for some Λ and a in X_Λ, $\omega_\Lambda(a) = 0$, then there exist minimal n, m in $\mathbb{N}$, such that $\omega \in \Omega^{n,m}$.

Proof. If 1_Λ^a is the characteristic function for the event $x_\Lambda = a$, it follows by invariance of ω and by the assumption that $\omega_\Lambda(a) = 0$

$$\int \mathrm{d}\omega\ (x) \sum_{k,l}{}' c(k, l; x)[(1_\Lambda^a)(x_{kl}) - (1_\Lambda^a)(x)]$$

$$= \int \mathrm{d}\omega\ (x) \sum_{k,l}{}' c(k, l; x)(1_\Lambda^a)(x_{kl}) = 0.$$

In particular, it follows for all $\{k, l\} \subset \Lambda$, with k and l, NN that $\omega_\Lambda(a_{kl}) = 0$. Repeating the argument we find that $\omega\{x \mid |x_\Lambda| = |a|\} = 0$. We shall say that $|a|$ is a forbidden cardinal in Λ.

By compatibility it follows that for all Δ with $\Lambda \subset \Delta$, and all a' in X_Δ with $a'_\Lambda = a : \omega_\Delta(a') = 0$, and therefore, again using stationarity: $\omega\{x \mid |a| \leqslant |x_\Delta| \leqslant |a| + (|\Delta| - |\Lambda|)\} = 0$.

If on the other hand $\Lambda \subset \Delta_1 \subset \Delta_2$, we have

$$\{x \mid |a| \leqslant |x_{\Delta_1}| \leqslant |a| + (|\Delta_1| - |\Lambda|)\} \subset \{x \mid |a| \leqslant |x_{\Delta_2}| \leqslant |a| + (|\Delta_2| - |\Lambda|)\}$$

so that

$$\omega\Big[\bigcup_{\Delta \supset \Lambda} \{x \mid |x_\Delta| \geqslant |a| \text{ and } |\bar{x}_\Delta| \geqslant |\Lambda| - |a|\}\Big] = 0$$

or

$$\omega\{x \mid |x| \geqslant |a| \text{ and } |\bar{x}| \geqslant |\Lambda| - |a|\} = 0.$$

Therefore $\omega \in \Omega^{|a|, |\Lambda|-|a|}$.

If $\{\Lambda_\alpha\}$ is an increasing sequence of rectangles, tending to $\mathbb{Z}^2$, let $n_\alpha^{(1)} < n_\alpha^{(2)} < \ldots < n_\alpha^{(k_\alpha)}$ be the forbidden cardinals in Λ_α. For $o < \beta$, we have

$$n_\beta^{(1)} \leqslant n_\alpha^{(1)} \quad \text{and} \quad |\Lambda_\beta| - n_\beta^{(k_\beta)} \leqslant |\Lambda_\alpha| - n_\alpha^{(k_\alpha)}.$$

Let $n = \inf_\alpha \{n_\alpha^{(1)}\}$ and $m = \inf_\alpha\{|\Lambda_\alpha| - n_\alpha^{(k_\alpha)}\}$. It follows that $\omega \in \Omega^{n,m}$. To prove uniqueness, let (n', m') be another couple determined along a sequence $\Lambda_{\alpha'}$ with $n' < n$. Choose α' large enough so that $n' = n_{\alpha'}^{(1)}$. A configuration $x_{\alpha'}$ with n' particles in $\Lambda_{\alpha'}$ is forbidden. Choose α large enough so that Λ_α covers $\Lambda_{\alpha'}$, and extend $x_{\alpha'}$ to a configuration x_α in Λ_α by defining $x_{\Lambda_\alpha \setminus \Lambda_{\alpha'}}$ to be identically zero. Then as $n_\alpha^{(1)} \geqslant n > n'$, $\omega_{\Lambda_\alpha}(x_\alpha) \neq 0$. But this is a contradiction, for $\omega_{\Lambda_\alpha}(x_\alpha) \leqslant \omega_{\Lambda_{\alpha'}}(x_{\alpha'}) = 0$. If $n < n'$ we can reverse the argument and conclude $n = n'$. A similar argument, counting vacancies, leads to $m' = m$. □

The rest of the analysis relies heavily on a treatment of the local specific free energy, and the basic lemma 1.23 in Holley and Stroock (1977).

If it is known that $\forall x_\Lambda \in X_\Lambda$, $\omega_\Lambda(x_\Lambda) > 0$ we define the local specific free energy in the state ω, denoting $H_\Lambda(x_\Lambda, 0)$ by $U_\Lambda(x_\Lambda)$

$$f_\Lambda(\omega) = \sum_{x_\Lambda \in X_\Lambda} [U_\Lambda(x_\Lambda) + (1/\beta) \ln \omega_\Lambda(x_\Lambda)]\omega_\Lambda(x_\Lambda).$$

If ω is a state as in the lemma of §3.1, we shall consider the reduced specific free energy

$$f_\Lambda^{n,m}(\omega) = \sum_{x_\Lambda \in X_\Lambda^{n,m}} [U_\Lambda(x_\Lambda) + (1/\beta) \ln \omega_\Lambda(x_\Lambda)]\omega_\Lambda(x_\Lambda).$$

Recalling that the configuration $a, (a_k)(k)$ in $X_{\Lambda \cup \{l\}}$, takes the value a in Λ and $1 - a(k)$ in l, let us introduce the values (for $a \in X_\Lambda$)

$$\Gamma_\Lambda(k, l; a) = \int \omega_{\tilde\Lambda}(a; \mathrm{d}x_{\tilde\Lambda}) c(k, l; a, x_{\tilde\Lambda})$$

$$\tilde\Gamma_{\Lambda l}(k, l; a) = \Gamma_{\Lambda \cup \{l\}}(k, l; a, (a_k)(k))$$

and, when $k, l \in \Lambda$:

$$V_\Lambda(k, l; a) = U_\Lambda(a_{kl}) - U_\Lambda(a)$$

$$V_\Lambda(a) = U_\Lambda(a_k) - U_\Lambda(a).$$

The next lemma adapts lemmas 1.10 and 1.16 in Holley and Stroock (1977) in that it takes into account particle transfer across the boundaries. We indicate the main steps only.

3.2. Lemma

If ω is T_t stationary, and for all Λ, all $a \in X_\Lambda$: $\omega_\Lambda(a) > 0$, then there exists a finite K, volume independent, such that

$$\sum_{k,l\in\Lambda}' \sum_{a\in X_\Lambda} 1/\beta(\Gamma_\Lambda(k,l;a)-\Gamma_\Lambda(k,l,a_{kl}))\ln\frac{\Gamma_\Lambda(k,l;a)}{\Gamma_\Lambda(k,l;a_{kl})}$$

$$+\sum_{\substack{k\in\Lambda\\ l\in\tilde\Lambda}}' \sum_{a\in X_\Lambda} 1/\beta(\tilde\Gamma_{\Lambda l}(k,l;a)-\tilde\Gamma_{\Lambda l}(k,l,a_k))\ln\frac{\tilde\Gamma_{\Lambda l}(k,l;a)}{\tilde\Gamma_{\Lambda 1}(k,l,a_k)}$$

$$\leqslant K\sum_{\substack{k,l\\ \{k,l\}\subset\partial\Lambda}}' \sum_{a\in X_\Lambda} |\Gamma_\Lambda(k,l,a)-\Gamma_\Lambda(k,l;a_{kl})|$$

$$+K\sum_{\substack{k,l\\ k\in\Lambda,l\in\tilde\Lambda}}' \sum_{a\in X_\Lambda} |\tilde\Gamma_{\Lambda l}(k,l;a)-\tilde\Gamma_{\Lambda l}(k,l;a_k)|.$$

Here $\partial\Lambda$ denotes the set of couples $\{k,l\}$ in Λ with k or l interacting with $\tilde\Lambda$.

Proof. As in step (3.47) in Georgii (1979) we may write, under the assumption $\omega_\Lambda(a)>0$ for all a in X_Λ:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}f_\Lambda(T_t^*\omega)\right|_{t=0}$$

$$=\sum_{k,l}'\int\omega(\mathrm{d}x)c(k,l;x)[U_\Lambda(x_{kl})-U_\Lambda(x)]$$

$$+1/\beta\sum_{k,l}'\sum_{a\in X_\Lambda}\ln\omega_\Lambda(a)\int\omega(\mathrm{d}x)c(k,l;x)[1_\Lambda^a(x_{kl})-1_\Lambda^a(x)]$$

$$=\sum_{k,l\in\Lambda}'\sum_{a\in X_\Lambda}\Big([U_\Lambda(a_{kl})-U_\Lambda(a)]\int\omega_{\tilde\Lambda}(a,\mathrm{d}x_{\tilde\Lambda})c(k,l;ax_{\tilde\Lambda})$$

$$+1/\beta\ln\omega_\Lambda(a)\int\omega(\mathrm{d}x)c(k,l;x)[1_\Lambda^{a_{kl}}(x)-1_\Lambda^a(x)]\Big)$$

$$+\sum_{\substack{k\in\Lambda\\ l\in\tilde\Lambda}}'\int\omega(\mathrm{d}x)c(k,l;x)\{[(U_\Lambda(x_{kl})-U_\Lambda(x))]$$

$$+1/\beta[\ln\omega_\Lambda((x_{kl})_\Lambda)-\ln\omega_\Lambda(x_\Lambda)]\}.$$

By a change of variables $a_{kl}\leftrightarrow a$, this equals

$$\sum_{k,l\in\Lambda}'\sum_{a\in X_\Lambda}[U_\Lambda(a_{kl})-U_\Lambda(a)+(1/\beta)\ln(\omega_\Lambda(a_{kl})/\omega_\Lambda(a))]\Gamma_\Lambda(k,l;a)$$

$$+\sum_{k\in\Lambda,l\in\tilde\Lambda}'\sum_{a\in X_\Lambda}\sum_{b=0,1}[U_\Lambda(a_{\Lambda\setminus\{k\}}b)-U_\Lambda(a)$$

$$+(1/\beta)\ln(\omega_\Lambda(a_{\Lambda\{k\}}b)/\omega_\Lambda(a))]\Gamma_{\Lambda\cup\{l\}}(k,l;ab).$$

In the last sum, we may restrict the b-summation to b different from $a(k)$. Hence we obtain, similarly to lemma 1.10 in Holley and Stroock (1977):

$$\tfrac{1}{2}\sum_{k,l\in\Lambda}'\sum_{a\in X_\Lambda}[V_\Lambda(k,l;a)-(1/\beta)\ln(\omega_\Lambda(a)/\omega_\Lambda(a_{kl}))]$$

$$\times[\Gamma_\Lambda(k,l;a)-\Gamma_\Lambda(k,l;a_{kl})]$$

$$+\tfrac{1}{2}\sum_{k\in\Lambda,l\in\tilde{\Lambda}}{}' \sum_{a\in X_\Lambda} [U_\Lambda(a_k)-U_\Lambda(a)-(1/\beta)\ln(\omega_\Lambda(a_k)/\omega_\Lambda(a)]$$
$$\times[\tilde{\Gamma}_{\Lambda l}(k,l;a)-\tilde{\Gamma}_{\Lambda l}(k,l;a_k)]$$
$$=-\tfrac{1}{2}\sum_{k,l\in\Lambda}{}' \sum_{a\in X_\Lambda}[\Gamma_\Lambda(k,l;a)-\Gamma_\Lambda(k,l;a_{kl})](1/\beta)\ln[\Gamma_\Lambda(k,l;a)/\Gamma_\Lambda(k,l;a_{kl})]$$
$$+\tfrac{1}{2}\sum_{k,l\in\Lambda}{}' \sum_{a\in X_\Lambda}[\Gamma_\Lambda(k,l;a)-\Gamma_\Lambda(k,l;a_{kl})]$$
$$\times[V_\Lambda(k,l;a)+(1/\beta)\ln(\Gamma_\Lambda(k,l;a)/\omega_\Lambda(a))$$
$$-(1/\beta)\ln(\Gamma_\Lambda(k,l;a_{kl})/\omega_\Lambda(a_{kl}))]-\tfrac{1}{2}\sum_{k\in\Lambda,l\in\tilde{\Lambda}}{}' \sum_{a\in X_\Lambda}[\tilde{\Gamma}_{\Lambda l}(k,l;a)$$
$$-\tilde{\Gamma}_{\Lambda l}(k,l;a_k)](1/\beta)\ln[\tilde{\Gamma}_{\Lambda l}(k,l;a)/\tilde{\Gamma}_{\Lambda l}(k,l;a_k)]$$
$$+\tfrac{1}{2}\sum_{k\in\Lambda,l\in\tilde{\Lambda}}{}' \sum_{a\in X_\Lambda}[\tilde{\Gamma}_{\Lambda l}(k,l;a)-\tilde{\Gamma}_{\Lambda l}(k,l;a_k)]$$
$$\times[V_\Lambda(k,a)+(1/\beta)\ln(\tilde{\Gamma}_{\Lambda l}(k,l;a)/\omega_\Lambda(a))$$
$$-(1/\beta)\ln(\tilde{\Gamma}_{\Lambda l}(k,l;a_k)/\omega_\Lambda(a_k))].$$

Next, it is easily checked that, when $\{k, l\}$ is a pair of nearest neighbours in Λ, not interacting with sites in $\tilde{\Lambda}$, the terms in the second sum vanish. Furthermore, a finite K, Λ independent, may be found, bounding the remaining terms

$$|V_\Lambda(k,l;a)+(1/\beta)\ln[\Gamma_\Lambda(k,l;a)/\omega_\Lambda(a)]-(1/\beta)\ln[\Gamma_\Lambda(k,l;a_k)/\omega_\Lambda(a_{kl})]|$$

and the corresponding terms in the last summation. Finally, we obtain by invariance of ω, the required estimation. Once again see Holley and Stroock (1977) for the details. □

In a completely analogous manner we obtain the corresponding result for stationary states in $\Omega^{n,m}$.

3.3. Lemma

If ω in $\Omega^{n,m}$ is stationary for T_t, then there exists a finite K', volume independent such that an estimation as in lemma 3.2. holds, with all summations restricted to a in $X_\Lambda^{n,m}$. □

3.4. Lemma

A state ω is T_t stationary iff, for all f, g in A and all $\{k, l\}\subset \mathbb{Z}^2$:

$$\int \omega(\mathrm{d}x)c(k,l;x)f(x_{kl})=\int \omega(\mathrm{d}x)c(k,l;x)f(x).$$

Proof. Due to inequality (1) or its counterpart of lemma 3.3, we can apply lemma 1.23 in Holley and Stroock (1977) and deduce for all Λ, all a in X_Λ (or $X_\Lambda^{n,m}$) all $\{k, l\}$ nearest neighbours in Λ that $\Gamma_\Lambda(k,l;a)=\Gamma_\Lambda(k,l;a_{kl})$.

By the density of the characteristic functions 1_Λ^a, the result follows for arbitrary f. By a transitivity argument the result is true for all k and l. □

At this point it should perhaps be remarked that we are not allowed to appeal to the result of theorem 2.14 Georgii (1979) as some $\omega_\Lambda(a)$ might vanish. To circumvent the difficulty, let us first show that it is sufficient to determine the extremal stationary states.

3.5. Lemma

The set of T_t-stationary states in a Choquet simplex.

Proof. Let ρ be $\in \mathcal{J}$, i.e. the T_t stationary self-adjoint continuous linear functionals on A. Write $\rho = \rho_1 - \rho_2$, with ρ_i positive linear functionals such that $\|\rho\| = \|\rho_1\| + \|\rho_2\|$. Then $\|T_t\rho_1 - T_t\rho_2\| = \|T_t\rho\| = \|\rho\| = \|\rho_1\| + \|\rho_2\| = \|T_t\rho_1\| + \|T_t\rho_2\|$. Hence $T_t\rho_1$ and $T_t\rho_2$ are orthogonal, and by uniqueness of the Jordan decomposition: $T_t\rho_1 = \rho_1$ and $T_t\rho_2 = \rho_2$. Therefore $T_t(|\rho|) = |\rho|$, and $\mathcal{J}$ is a lattice with the w^* compact convex set of stationary states as a basis. □

3.6. Proposition

If ω is an extremal T_t-stationary state, then ω is an extremal canonical Gibbs state.

Proof. Any L^∞ function g for which $g(x_{kl}) = g(x)$ for all k and l in Z^2, is ω almost everywhere constant, for if not, the functional $\omega(g\,\cdot\,)$ is stationary by lemma 3.4. and thus ω is not extremal stationary.

Hence $\omega\{x|\,|x| = +\infty$ and $|\bar{x}| = +\infty\}$ is 0 or 1. In the former case it follows, again by extremality and the fact that the events $\{x|\,|x| = n\}$, $\{x|\,|\bar{x}| = m\}$ are permutation invariant, that there exists an n_0 or m_0 such that $\omega\{x|\,|x| = n_0\} = 1$ or $\omega\{x|\,|\bar{x}| = m_0\} = 1$. It then follows from lemma 3.4, assuming the first situation prevails with $n_0 > 0$:

$$\omega(x_\Lambda = a/E_\Lambda) = \exp[-\beta H_\Lambda(a,\cdot\,)]\chi^{n_0}(a,\cdot\,)\Bigg(\sum_{\substack{b\in X_\Lambda\\|b|=|x_\Lambda|}} \exp[-\beta H_\Lambda(b,\cdot\,)]\chi^{n_0}(b,\cdot\,)\Bigg)^{-1}.$$

But by the martingale convergence theorem, and extremal stationarity of ω, it follows for all Δ:

$$\omega\{x|\omega_\Delta(a) = \lim_{\Lambda\uparrow Z^2} \omega(x_\Delta = a/E_\Lambda) \quad \text{for all } a \in X_\Delta\cdot\} = 1.$$

Choose $|a| = n_0$, then for $\Delta \subset \Lambda$, and o the empty configuration on $\Lambda\backslash\Delta$:

$$\lim_{\Lambda\uparrow Z^2} \omega(x_\Delta = a/E_\Lambda) = \lim_{\Lambda\uparrow Z^2} \sum_{c\in X_\Lambda} 1_\Delta^a(c)\omega(x_\Lambda = c/E_\Lambda)$$

$$= \lim_{\Lambda\uparrow Z^2} \omega(x_\Lambda = a\mathrm{o}/E_\Lambda)$$

$$= \lim_{\Lambda\uparrow Z^2} \frac{\exp[-\beta H_\Lambda(a\mathrm{o},\cdot\,)]\chi^{n_0}(a\mathrm{o},\cdot\,)}{\sum_{b\in X_\Lambda,|b|=n_0} \exp[-\beta H_\Lambda(b,\cdot\,)]\chi^{n_0}(b,\cdot\,)} \text{AE}.$$

If we let $\alpha = \min_{b \in X_\Lambda, |b| = n_0}\{\exp[-\beta H_\Lambda(b, 0)]\} > 0$ then

$$\omega(x_\Lambda = ao/E_\Lambda) \leq \frac{\exp[-\beta H_\Lambda(ao, \cdot)]\chi^{n_0}(ao, \cdot)n_0!}{\alpha \cdot |\Lambda|(|\Lambda|-1)\ldots(|\Lambda|-n_0+1)}.$$

Hence for all Δ, all $a \in X_\Delta$ with $|a| = n_0$, we have $\omega_\Delta(a) = 0$, which is a contradiction. A similar argument holds when $\omega\{x | |\bar{x}| = m_0\} = 1$. We therefore conclude that either $\omega = \delta_0$ or δ_i, or ω is a canonical Gibbs state (with an infinity of particles and vacant sites), following part 2 of proposition 2.19 of Georgii (1979). But since, on the contrary, any canonical Gibbs state is stationary, we conclude that ω is an extremal canonical Gibbs state. □

4. Return to equilibrium

We finally add a brief remark on the return to equilibrium described by the dynamics. By this we mean that any state ρ, absolutely continuous with respect to a canonical Gibbs state ω, $\lim_{t\to\infty} T_t^* \rho$ exists in the norm topology. As emphasised by Davies (1977) the norm condition discriminates real dissipation of local disturbances from their migration to infinity.

The semigroup defined in § 2.4. may be extended to a self-adjoint contraction semigroup on $L^2(X, d\omega)$ as in lemma 1.3 of Holley and Stroock (1976) for example. Then as T_t is positivity preserving, it is extendable to an L^p contractive semigroup, following theorem X.55 in Reed and Simon (1975). In particular it is an L^1 strongly continuous semigroup, and by duality the semigroup restricted to $L^\infty(X, d\omega)$ is a dynamical semigroup in the sense that T_t is $\sigma(L^\infty, L^1)$ continuous for all t, and that the map $t \to T_t f$ is $\sigma(L^\infty, L^1)$ continuous.

The above statement on the return to equilibrium, and the fact that the limiting state of $T_t^* \rho$ is $\rho \cdot E$, with E the unique T_t invariant conditional expectation on the T_t invariant functions in $L^\infty(X, d\omega)$, may then be seen to be consequences of different arguments. In a direct way they follow from Rota's theorem (Rota 1962, Doob 1963), or theorem 4.2. in Frigerio (1978); indirectly one may first prove $L^2(X, d\omega)$ convergence as in Holley and Stroock (1976) and then proceed via density arguments.

Of course the above result cannot describe the physically very interesting processes of cooling the gas from an equilibrium state at very high temperatures, to a gas–liquid system in equilibrium at a temperature below the critical one. More refined techniques should be used to (dis)prove the convergence of states, not absolutely continuous with respect to a given stationary state, under the semigroup, to that state. This convergence is generally called 'approach to equilibrium'.

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